Dominating Sets in Directed Graphs

Chaoyi Pang [a,*] Rui Zhang [b] Qing Zhang [a] Junhu Wang [c]

[a] *The Australian e-Health Research Centre, ICT Centre, CSIRO, Australia*

[b] *Department of Computer Science and Software Engineering, The University of Melbourne, Australia*

[c] *School of Information and Communication Technology, Griffith University, Australia*

Abstract

We consider the problem of incrementally computing a minimal dominating set of a directed graph after the insertion or deletion of a set of arcs. Earlier results have either focused on the study of the properties that minimum (not minimal) dominating sets preserved or lacked to investigate which update affects a minimal dominating set and in what ways. In this paper, we first show how to incrementally compute a minimal dominating set on arc insertions. We then reduce the case of computing a minimal dominating set on arc deletions to the case of insertions. Some properties on minimal dominating sets are provided to support the incremental strategy. Lastly, we give a new bound on the size of minimum dominating sets based on those results.

Key words: Directed graph (digraph), Minimal/minimum dominating set, Out-domination, In-domination

1 Introduction

Whilst domination in undirected graphs have been studied extensively [5], domination in directed graphs (digraphs) have not yet gained the same amount of attention from researchers. For domination in digraph, the research of [1,2,4] study the properties of the minimum dominating sets upon some characterized digraphs. More recently, domination in digraphs has been used in the study of answering skyline query in database [6] and routing in networks [11]. A classic

* Corresponding author.
Email address: chaoyi.pang@csiro.au (Chaoyi Pang).
URL: http://www.ict.csiro.au/staff/chaoyi.pang (Chaoyi Pang).

Preprint submitted to Elsevier *8 June 2010*

application example is to choose a hotel from many candidates by considering two parameters in making decision, price and distance to the beach, . Quite often there is no hotel that satisfies both objectives, i.e., with minimized price and minimized distance to the beach, because hotels close to the beach tend to have higher prices. The skyline query returns all the interesting hotels, which are the ones that are no worse than any other hotel for both parameters. Formally, these are the hotels not dominated by any other one. If we view every parameter as a dimension, then p dominates q if p is no worse than q in every dimension and better than q in at least one dimension. By adding other preferences such as facilities and meal prices, this example is extended into high dimensions which can be modeled by the domination problem in digraphs generally [10]. Recently, the researches on skyline are extended to process spatial data [9] and stream data [7,12].

On a large data set where the updates are small and frequent, the construction of the new answer from the old answer should be more efficient and responsive than recomputing the new answer from scratch [3]. In database research, incremental computing queries or graph problems have gained great attentions recently [8].

In this paper, we study the variations of a dominating set upon updating digraphs. We investigate how to compute incrementally a minimal dominating set on digraphs and show how a single update affects a dominating set. From this study, a new bound on the size of minimum dominating set is derived.

The rest of the paper is organized as follows. Section 2 explains the basic terminologies. Section 3 presents the results of incremental computability and provides the related properties. Section 4 concludes this paper.

2 Dominating Set

A *directed graph* (digraph) is a pair $G = (V, A)$, where V is a set of nodes and $A \subseteq V \times V$ is a set of *arcs*, i.e., ordered pairs of nodes. We say that node u dominates node v (or node v is dominated by node u) if arc (u, v) is in G. Arc (u, v) is denoted as $u \to v$ graphically. In digraph G, the *indegree* of node v is the number of arcs directed into v and is denoted as $d^-_G(v)$, i.e., $d^-_G(v) = |\{(u, v)|(u, v) \in G\}|$. The *outdegree* of node u is the number of arcs going out of u and is denoted as $d^+_G(u)$, i.e., $d^+_G(u) = |\{(u, v)|(u, v) \in G\}|$.

A new arc can be added into G or an existing arc can be removed from G. We denote the new graph $G_{+e} = (V, A \cup \{e\})$ when a new arc $e \in V \times V - A$ is added into G. Similarly, we denote the new graph $G_{-f} = (V, A - \{f\})$ when an arc $f \in A$ is deleted from G.

In digraph $G = (V, A)$, a set of nodes $S \subseteq V$ is a *dominating set* [1] of G if each node $v \in V - S$ is dominated by at least a node in S. A *minimal* dominating set S_m is a dominating set with no proper subset of S_m as a dominating set. Suppose that u is a node of dominating set S. Node u is called *redundant* if $S_{-u} = S - \{u\}$ is still a dominating set. A *minimum* dominating set S_M is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$. Clearly, a node u with $d^-(u) = 0$ belongs to every S_m and S_M.

3 Constructing a minimal dominating set incrementally

In this section, we first study how S, a minimal dominating set of $G = (V, A)$, changes when inserting a new arc into G. We show that a minimal dominating set of G_{+e} can be derived from S in a simple way. We then apply the result on insertions to the case of arc deletions and derive a reduction result (Theorem 2). The reduction result shows that a minimal dominating set of G_{-f} can be derived through arc insertion operations.

The following lemma will be used to prove the major results in this section. Intuitively, the following lemma means that (1) any dominating set of a subgraph of G is a dominating set of G, and (2) any minimal dominating set of G is a minimal dominating set a subgraph of G if and only if it is a dominating set of the subgraph.

Lemma 1 *Suppose S and S' be a minimal dominating set of $G = (V, A)$ and $G' = (V, A')$ respectively. If $A' \subseteq A$, then*

(1) S' is a dominating set of G.
(2) S is a minimal dominating set of G' if S is a dominating set of G'.

Proof: The proof of (1): Since S' is a dominating set of G' and $A' \subseteq A$,

$$\begin{aligned} V - S' &\subseteq \{w | (s, w) \in A' \cap [S' \times (V - S')]\} \\ &\subseteq \{w | (s, w) \in A \cap [S' \times (V - S')]\} \end{aligned}$$

holds. The above equation means that S' dominates each node of $V - S'$ in G. Thus, S' is a dominating set of G by the definition of dominating set.

The proof of (2): Otherwise, S is a dominating set but not a minimal dominating set of G'. Then $S - \{s\}$ is still a dominating set of G' for certain node

[1] It is called out-dominating set in [2].

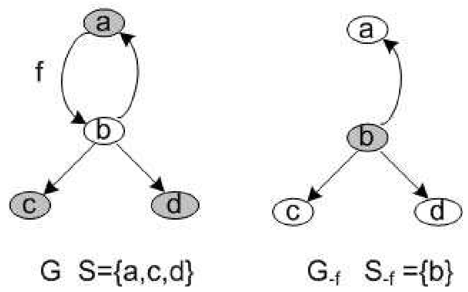

Fig. 1. S and S_{-f} are the minimal dominating sets of G and G_{-f} respectively.

$s \in S$. Consequently, $S - \{s\}$ is a dominating set of G by Lemma 1(1), which is contradictory to the hypothesis that S is a minimal dominating set of G. □

Roughly, Lemma 1(1) indicates that a minimal dominating set of G_{+e} is very similar to a minimal dominating set of G and can be generated from it cheaply. In fact, as indicated in Theorem 1 afterwards, there exist S and S_{+e}, minimal dominating sets of G and G_{+e} respectively, such that $|S \cup S_{+e} - S \cap S_{+e}| \leq 1$ holds. However, this property does not hold for the minimal dominating sets of G and G_{-f}. That is, given a minimal dominating set S of G, there may not exist a minimal dominating set S_{-f} of G_{-f} that is close to S in size. The following is a counter-example for the decremental case.

Example 1 *Let G and G_{-f} be the digraphs in Figure 1 for $f = (a, b)$. It can be proven that $S = \{a, c, d\}$ is a minimal dominating set of G and $S_{-f} = \{b\}$ is the unique minimal (or minimum) dominating set of G_{-f}. It concludes that S_{-f} is a dominating set of G by Lemma 1(1). Clearly, S_{-f} is a minimal dominating set of G as $|S_{-f}| = 1$.*

Instinctively, this result indicates that a minimal dominating set of G_{-f} is quite "similar" to a minimal dominating set of G. That is, the generation of a minimal dominating set of G via a minimal dominating set of a subgraph of G can be very efficient. This will be explored in Theorem 1.

In contrast, since $|S \cup S_{-f} - S \cap S_{-f}|$ equals the number of nodes in the graph $|V|$, this implies that deriving S_{-f} from S may require the removal of many nodes from S. That is, since the difference of S and S_{-f} can be immense, it could be more expensive (in time) to generate S_{-f} from S in comparison to the derivation of S from S_{-f}. This will be evidenced in Theorem 2.

Furthermore, as indicated in Corollary 1 and analogous to Theorem 1, the sizes of minimum dominating sets of G_{-f}, G and G_{+e} are rather close.

3.1 A minimal dominating set of G_{+e}

Below is the detailed explanation on constructing a minimal dominating set of G_{+e} from a minimal dominating set of G.

Let S be a minimal dominating set of $G = (V, A)$ and $e = (x, y) \notin A$. Since $G \subset G_{+e}$ and according to Lemma 1(1), S is a dominating set of G_{+e}. To derive a minimal dominating set of G_{+e} from S, we have the following two cases: (1) $x \notin S$ and (2) $x \in S$.

(1) If $x \notin S$, we will claim that S is a minimal dominating set of G_{+e} by the definition of a minimal dominating set.

Since S is a minimal dominating set of G, then for each $s \in S$ there exists node $w \in (V - S) \cup \{s\}$ such that w cannot be dominated by any nodes of $S - \{s\}$ in G. Since no new arcs from $S - \{s\}$ to w is inserted considering that $e = (x, y)$ and $x \notin S$, this node w cannot be dominated by any nodes of $S - \{s\}$ in G_{+e} either. Therefore, S is a minimal dominating set of G_{+e} from the definition of a minimal dominating set.

Therefore, for a given minimal dominating set S of $G = (V, A)$ and arc $e = (x, y) \notin A$, the above proved that S is a minimal dominating set of G_{+e} if $x \notin S$.

(2) If $x \in S$, we will claim that S may not be a minimal dominating set of G_{+e}. To derive a minimal dominating set of G_{+e} from S under this situation, we need to consider the two subcases: (i) $y \in S$ and (ii) $y \in V - S$.

(i) If nodes $x, y \in S$, the insertion of e might make y redundant *only if* each node of $\{w|(y, w) \in A, w \notin S\}$ can be dominated by a node of $S - \{y\}$ (Figure 2(i)). That is,

$$\{w|(y, w) \in A, w \notin S\} \subseteq \{v|(s, v) \in A, s \in S_{-y}\} \tag{1}$$

where $S_{-y} = S - \{y\}$.

Moreover, if S_{-y} is a dominating set but not a minimal dominating set of G_{+e}, then there exists $s' \in S_{-y}$ such that $S_{-y} - \{s'\}$ is a dominating set of G_{+e}. This will result in $S - \{s'\}$ being a dominating set of G which is contradictory to the fact that S is a minimal dominating set of G. Therefore, $S - \{y\}$ is a minimal dominating set of G_{+e}.

Similarly, it can be proven that S is a minimal dominating set of G_{+e} if Formula (1) does not hold.

(ii) If node $x \in S$ but node $y \in (V - S)$, the insertion of e might cause a node $u \in S$ that dominates node y in G to be redundant. Let $S_{-u} = S - \{u\}$. A node $u \in S$ is redundant if the following formula holds:

$$\begin{aligned}&(u, s \in S) \wedge (u \neq s) \wedge ((s, u), (u, y) \in A) \wedge \\ &[\{v|(u, v) \in A, v \in V - S \cup \{y\}\} \subseteq \{w|(s', w) \in A, s' \in S_{-u}\}].\end{aligned} \tag{2}$$

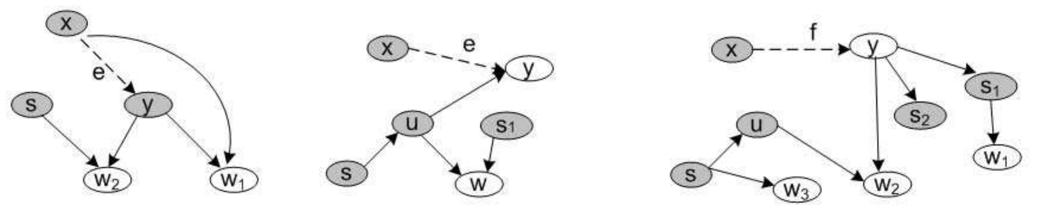

Fig. 2. S, S_{+e} and S_{-f} are minimal dominating sets of G, G_{+e} and G_{-f} respectively.

That is, each node that dominated by u is also dominated by other node in S as depicted in Figure 2(ii). It should be noted that such node u is *unique* if it exists (otherwise, it will be contradictory to the assumption that S is minimal). In this situation, it can be proven that $S-\{u\}$ is a minimal dominating set of G_{+e}. Similarly, if there does not exist such node u satisfying Formula (2), it can be proven that S is a minimal dominating set of G_{+e} as each $u \in S$ such that $(u, y) \in A$ is not redundant in G_{+e}.

Through summarizing the above, we have the following result on a single arc insertion.

Theorem 1 *Let S be a minimal dominating set of $G = (V, A)$ and suppose that $e = (x, y) \notin A$ where $x, y \in V$. Then*

$$S_{+e} = \begin{cases} S-\{y\} & \text{if } x,y \in S \text{ and Formula (1) holds;} \\ S-\{u\} & \text{if } ((x,y) \in S \times (V-S)) \text{ and Formula (2) holds for node } u; \\ S & \text{otherwise.} \end{cases}$$

is a minimal dominating set of $G_{+e} = (V, A \cup \{e\})$.

The above result says that, for any given minimal dominating set S of G, there exists a minimal dominating set S_{+e} of G_{+e} such that S_{+e} can be derived by removing at most one node from S. That is,

$$S - \{w\} \subseteq S_{+e} \subseteq S \tag{3}$$

holds for a node $w \in S$ and S_{+e} can be set to S if $e = (x, y)$ is in $(V - S) \times V$.

To use Theorem 1 on inserting a set of arcs I, a straightforward way is to divide I into two disjoint subsets I_1 and I_2; I_1 is the subset of I where $x \in S$ for each arc (x, y) in I and I_2 is the set of the rest arcs of I. By Theorem 1, S, the minimal dominating set of G, is a minimal dominating set of $G' = G_{+I_2}$. Thus, a minimal dominating set S_{+I} of G_{+I} can be constructed from S and G' rather than scratch built from S and G. This strategy can be more efficient on generating a minimal dominating set of G_{+I} as some operations on the arcs of I_2 can be avoided.

3.2 A minimal dominating set of G_{-f}

Let S be a minimal dominating set of $G = (V, A)$ and $f = (x, y) \in A$. As indicated in Figure 1, a minimal dominating set of G_{-f} can be very different from S. However, since it can easily prove that $S \cup \{y\}$ is a dominating set of G_{-f}, a minimal dominating set of G_{-f} can be derivable from $S \cup \{y\}$ and may require the removal of many S nodes. Below is the detailed explanation.

(1) S is a minimal dominating set of G_{-f} if $f \notin A \cap [S \times (V - S)]$. In this situation, S is a dominating set of G_{-f} as each node of $V - S$ is still dominated by a node of S in G_{-f}. Moreover, since $G_{-f} \subset G$ and according to Lemma 1(2), S is a minimal dominating set of G_{-f}.

(2) Otherwise, S may not be a minimal dominating set of G_{-f} if $f \in A \cap [S \times (V - S)]$. In this situation, the deletion of $f = (x, y)$ requires node y to be added into S if node y is not dominated by S in G_{-f} any more. Sequentially, this may result in some other nodes of S to become redundant. Specifically, from Lemma 1(2), S is a minimal dominating set of G_{-f} if

$$\{w|(w, y) \in A_{-f}\} \cap S \neq \emptyset \tag{4}$$

holds for $A_{-f} = A - \{f\}$. Or else, $S \cup \{y\}$ is a dominating set of G_{-f} but may not be a minimal.

In order to derive a minimal dominating set S' of G_{-f} from $S \cup \{y\}$, some redundant nodes of $S \cup \{y\}$ in G_{-f} need to be removed as indicated in Example 1. In the following, we show that a minimal dominating set $S' \subseteq S \cup \{y\}$ of G_{-f} can be derived in terms of arcs insertion through using Theorem 1.

Let $G' = (V, A')$ where $A' = A - A \cap (\{y\} \times V) - \{f\}$. It can be proven that $S' = S \cup \{y\}$ is a minimal dominating set of G'. Therefore, a minimal dominating set of G_{-f} can be derived from $S \cup \{y\}$ through repeatedly inserting an arc of (i) $A \cap (\{y\} \times S)$ and (ii) $A \cap (\{y\} \times (V - S))$ into G'. Thus, according to Theorem 1, a redundant node of $S \cup \{y\}$ in G_{-f} can only be a node of

$$\{s|(y, s) \in A \cap (\{y\} \times S)\}, \tag{5}$$

or

$$\{s|(s, w) \in A \wedge (y, w) \in (\{y\} \times (V - S))\}. \tag{6}$$

The above discussion leads Theorem 2 on a single arc deletion.

Theorem 2 *Let S be a minimal dominating set of $G = (V, A)$ and $f =$*

$(x, y) \in A$. *Let denote* $I = A \cap (\{y\} \times V) - \{f\}$ *and* $G' = (V, A - I)$. *Then*

$$S_{-f} = \begin{cases} S' & \text{where } S' \subseteq S \cup \{y\} \text{ is a minimal dominating set of } G'_{+I} \\ & \text{if } (x,y) \in S \times (V - S) \text{ and } \{w|(w,y) \in A_{-f}, w \in S\} = \emptyset \text{ hold;} \\ S & \text{otherwise.} \end{cases}$$

is a minimal dominating set of $G_{-f} = (V, A - \{f\})$.

Example 2 *We illustrate Theorem 2 with Figure 2(iii). Since* $\{w|(w, y) \in A_{-f}\} = \emptyset$ *(i.e., Formula (4) does not hold),* $S' = S \cup \{y\}$ *is a dominating set of* G_{-f}. *By Formula (5) and Formula (6),*

$$\{s|(y, s) \in A \cap (\{y\} \times S)\} = \{s_1, s_2\},$$

and

$$\{s|(s, w) \in A \wedge (y, w) \in (\{y\} \times (V - S))\} = \{u\}.$$

Let $A' = A - A \cap (\{y\} \times V) - \{f\} = \{(s, u), (s, w_3), (u, w_2), (s_1, w_1)\}$. *Then* S' *is a minimal dominating set of* $G' = (V, A')$ *as stated in the proof of Theorem 2. With* G' *and* S', *we will derive* S_{-f} *through inserting arcs of (i)* $A \cap (\{y\} \times S) = \{(y, s_1), (y, s_2)\}$ *and (ii)* $A \cap (\{y\} \times (V - S)) = \{(y, w_2)\}$ *into* G'. *From Theorem 1, the insertion of arc* (y, s_2) *makes node* s_2 *redundant and the insertion of arc* (y, w_2) *makes node* u *redundant. Therefore,* $S_{-f} = \{x, y, s, s_1, s_3\}$.

Similar to the situation of inserting a set of arcs, a straightforward application of Theorem 2 on the deletion of arc set D can be performed by first deleting D_2 and then deleting D_1; where D_1 is the set of arcs of $D \cap [S \times (V - S)]$ and $D_2 = D - D_1$. Theorem 2 ensures that S is a minimal dominating set of $G_{-D_2} = (V, A - D_2)$.

3.3 Size bound on the minimum dominating sets

Generally, as indicated in Example 1, it may not find a minimal dominating set S_{-f} of G_{-f} that is close to a given minimal dominating set S of G. However, we have the following interesting result for the minimum dominating sets of G, G_{+e} and G_{-f}.

Corollary 1 *Let* $G = (V, A)$ *be a digraph and* $f \in A$ *and* $e \in (V \times V) - A$. *Then*

$$\gamma(G_{+e}) \leq \gamma(G) \leq \gamma(G_{-f}), \quad (7)$$

$$\gamma(G_{-f}) - 1 \leq \gamma(G) \leq \gamma(G_{+e}) + 1 \quad (8)$$

hold.

Proof: Clearly, Equation (7) holds as each dominating set of G_{-f} is a dominating set of G and, consequently, is a dominating set of G_{+e}.

By Theorem 1, it derives that $\gamma(G) \leq \gamma(G_{+e})+1$ holds. Again from Theorem 1, it derives that $\gamma(G_{-f}) \leq \gamma(G) + 1$ or $\gamma(G_{-f}) - 1 \leq \gamma(G)$ holds. Therefore, Equation (8) holds. □

Different from those properties on the minimum dominating set such as those of [1,2,4], Corollary 1 describes the relation among $\gamma(G)$, $\gamma(G_{+e})$ and $\gamma(G_{-f})$ and shows they are very close in sizes.

Let S_M be a minimum dominating set of G. From Corollary 1, Theorem 2 can be simplified by using S_M. That is, there exists a minimal dominating set S_{-f} of G_{-f} such that $S_{-f} \subseteq S_M \cup \{y\}$ and S_{-f} can be derived by removing at most one node from $S_M \cup \{y\}$.

4 Conclusions

In this paper, we have investigated how a minimal dominating set of G varies in G_{+e} and G_{-f}. With the paper's results, a dominating set of an updated graph can be constructed from a dominating set of the original graph without overly processing the whole graph. We indicate that a minimal dominating set of G_{+e} can be efficiently derived from a minimal dominating set of G and both of which share a high level of similarity to each other. We also show the situation for decremental is different. That is, there may not exist a minimal dominating set of G_{-f} that is close to the size of a given minimal dominating set of G. However, a minimal dominating set of G_{-f} can be derived from a minimal dominating set of G through adding a node and deleting some "closer" nodes.

Our future work will evaluate the efficiency of this technique in reality. Furthermore, it is also interesting to study the approximation of the minimum dominating set of G through the use of Corollary 1.

Acknowledgments: The authors would like to thank the reviewers for their comments on a draft of this paper.

References

[1] G. Chartrand, P. Dankelmann, M. Schultz, and H. C. Swart. Twin domination in digraphs. *Ars Comb*, 67, 2003.

[2] G. Chartrand, F. Harary, and B. Q. Yue. On the out-domination and in-domination numbers of a digraph. *Discrete Mathematics*, 197-198, 1999.

[3] G. Dong and C. Pang. Maintaining transitive closure in first-order after node-set and edge-set deletions. *Information Processing Letters*, 62(3), 1997.

[4] J. Ghosal, R. Laskar, and D. Pillone. *Topics on domination in directed graphs, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[5] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[6] V. Koltun and C. H. Papadimitriou. Approximately dominating representatives. In *ICDT '05: Proceedings of International Conference on Data Theory*, 204–214, 2005.

[7] M. Morse, J. M. Patel, and W. I. Grosky. Efficient continuous skyline computation. *Inf. Sci.*, 177(17):3411–3437, 2007.

[8] C. Pang, G. Dong, and K. Ramamohanarao. Incremental maintenance of shortest distance and transitive closure in first-order logic and sql. *ACM Trans. Database Syst.*, 30(3):698–721, 2005.

[9] M. Sharifzadeh, C. Shahabi, and L. Kazemi. Processing spatial skyline queries in both vector spaces and spatial network databases. *ACM Trans. Database Syst.*, 34(3):1–45, 2009.

[10] W. T. Trotter. Dimension of the crown S_n^k. *Discrete Mathematics*, 8(1):85–103, 1974.

[11] J. Wu. Extended dominating-set-based routing in ad hoc wireless networks with unidirectional links. *IEEE Trans. Parallel Distrib. Syst.*, 13(9), 2002.

[12] Z. Zhang, R. Cheng, D. Papadias, and A. K. Tung. Minimizing the communication cost for continuous skyline maintenance. In *SIGMOD '09: Proceedings of the 35th SIGMOD international conference on Management of data*, pages 495–508, New York, NY, USA, 2009. ACM.